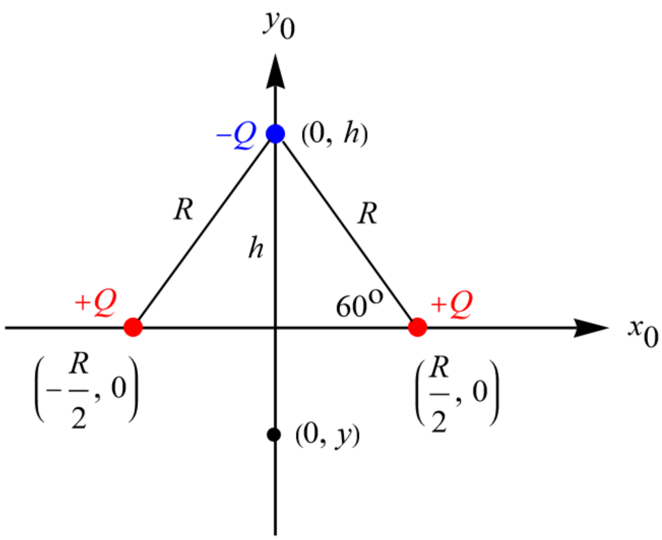
Problem 1.2

Zero force from a triangle

Two positive ions and one negative ion are fixed at the vertices of an equilateral triangle. Where can a fourth ion be placed, along the symmetry axis of the setup, so that the force on it will be zero? Is there more than one such place? You will need to solve something numerically.

Solution

Draw a schematic of the three ions at the vertices of an equilateral triangle.



Use trigonometry to determine h.

$$\sin 60^\circ = \frac{h}{R} \quad \rightarrow \quad h = R\sin 60^\circ = \frac{R\sqrt{3}}{2}$$

A fourth ion placed on the vertical axis will have zero force acting on it wherever the electric field is zero. Call this location $(0, y)$ and let $\boldsymbol{\mathscr{r}}_i$ be the position vector from charge q_i to $(0, y)$.

$$\begin{aligned}
\mathbf{0} &= \mathbf{E} \\
&= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i \\
&= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\frac{q_i}{\mathscr{r}_i^2}\left(\frac{\boldsymbol{\mathscr{r}}_i}{\mathscr{r}_i}\right) \\
&= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{3}\frac{q_i}{\mathscr{r}_i^3}\boldsymbol{\mathscr{r}}_i
\end{aligned}$$

Multiply both sides by $4\pi\epsilon_0$.

$$\mathbf{0} = \sum_{i=1}^{3}\frac{q_i}{\mathscr{r}_i^3}\boldsymbol{\mathscr{r}}_i$$

Write out the terms for each of the three charges.

$$\mathbf{0} = \frac{Q}{\left[\left(0+\frac{R}{2}\right)^2 + (y-0)^2\right]^{3/2}}\left[\left(0+\frac{R}{2}\right)\hat{\mathbf{x}}_0 + (y-0)\hat{\mathbf{y}}_0\right]$$
$$+ \frac{Q}{\left[\left(0-\frac{R}{2}\right)^2 + (y-0)^2\right]^{3/2}}\left[\left(0-\frac{R}{2}\right)\hat{\mathbf{x}}_0 + (y-0)\hat{\mathbf{y}}_0\right]$$
$$+ \frac{-Q}{[(0-0)^2 + (y-h)^2]^{3/2}}[(0-0)\hat{\mathbf{x}}_0 + (y-h)\hat{\mathbf{y}}_0]$$

Divide both sides by Q and combine like-terms.

$$\mathbf{0} = \frac{1}{\left(\frac{R^2}{4}+y^2\right)^{3/2}}\left(\frac{R}{2}-\frac{R}{2}+0\right)\hat{\mathbf{x}}_0 + \frac{1}{\left(\frac{R^2}{4}+y^2\right)^{3/2}}(y+y)\hat{\mathbf{y}}_0 - \frac{1}{|y-h|^3}(y-h)\hat{\mathbf{y}}_0$$
$$= \left[\frac{2y}{\left(y^2+\frac{R^2}{4}\right)^{3/2}} - \frac{y-\frac{R\sqrt{3}}{2}}{\left|y-\frac{R\sqrt{3}}{2}\right|^3}\right]\hat{\mathbf{y}}_0$$

The quantity in square brackets must be zero for this equation to hold.

$$\frac{2y}{\left(y^2+\frac{R^2}{4}\right)^{3/2}} - \frac{y-\frac{R\sqrt{3}}{2}}{\left|y-\frac{R\sqrt{3}}{2}\right|^3} = 0$$

Multiply both sides by $|y-h|^3/(y-h)$.

$$\frac{2y}{\left(y^2+\frac{R^2}{4}\right)^{3/2}}\frac{\left|y-\frac{R\sqrt{3}}{2}\right|^3}{y-\frac{R\sqrt{3}}{2}} - 1 = 0$$

Write the equation in terms of y/R, a dimensionless variable.

$$\frac{2y}{R^3\left(\frac{y^2}{R^2}+\frac{1}{4}\right)^{3/2}}\frac{R^3\left|\frac{y}{R}-\frac{\sqrt{3}}{2}\right|^3}{R\left(\frac{y}{R}-\frac{\sqrt{3}}{2}\right)} - 1 = 0$$

Simplify the left side

$$\frac{2\left(\frac{y}{R}\right)\left(\frac{y}{R}-\frac{\sqrt{3}}{2}\right)^2\operatorname{sgn}\left(\frac{y}{R}-\frac{\sqrt{3}}{2}\right)}{\left[\left(\frac{y}{R}\right)^2+\frac{1}{4}\right]^{3/2}} - 1 = 0$$

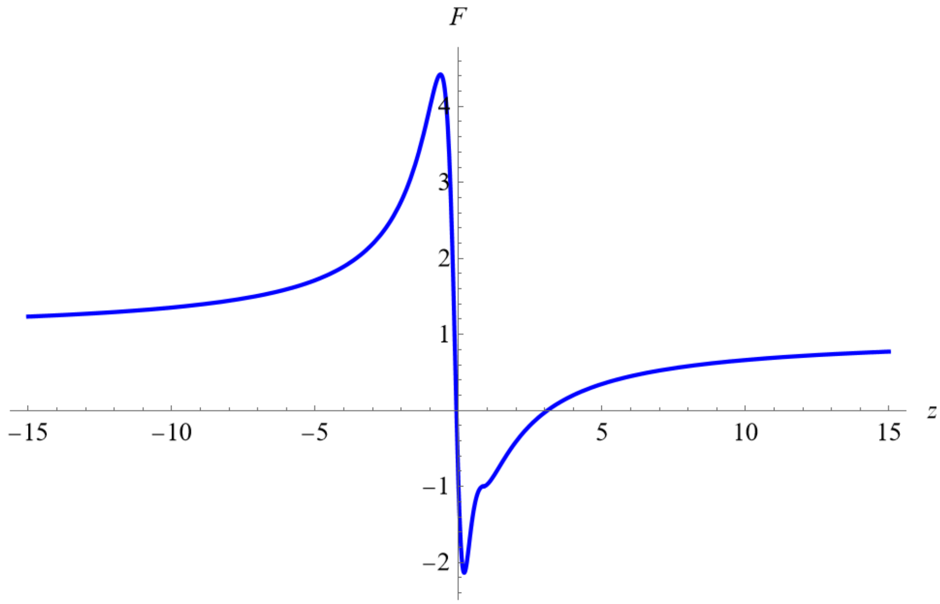
and call it $F(z)$, where $z = y/R$.

$$F(z) = 0$$

Solving this equation numerically yields two results.

$$z \approx \{-0.0731451, 3.10224\} \quad \Rightarrow \quad y \approx \{-0.0731451R, 3.10224R\}$$

Plotting $F(z)$ versus z and seeing where the graph crosses the horizontal axis confirms these numbers.



Therefore, a fourth ion can be placed at about $(0, -0.0731451R)$ and at about $(0, 3.10224R)$ in which it experiences zero force, where R is the distance between the equilateral triangle vertices.